The stronger mixing variables method

Pham Kim Hung

1. Stronger mixing variables - S.M.V theorem

The following result only uses elementary mathematics. Understanding the theorem's usage and its meaning is more important to you than remembering its detailed proof.

Lemma. (General mixing variables lemma). *Suppose that $(a_1, a_2, ..., a_n)$ is an arbitrary real sequence. Carry out the following transformation consecutively*

1. *Choosing $i, j \in \{1, 2, ..., n\}$ to be two indices satisfying*

$$a_i = \min(a_1, a_2, ..., a_n) \quad , \quad a_j = \max(a_1, a_2, ..., a_n).$$

2. *Replacing a_i and a_j by $\dfrac{a_i + a_j}{2}$ (but their orders don't change).*

After doing infinitely many of the above transformations, each number a_i comes to the same limit

$$a = \frac{a_1 + a_2 + ... + a_n}{n}.$$

Proof. Henceforward, the above transformation is called the Δ transformation. Denote the first sequence as $(a_1^1, a_2^1, ..., a_n^1)$. After one transformation, we have a new sequence, denoted as $(a_1^2, a_2^2, .., a_n^2)$. Similarly, from the sequence $(a_1^k, a_2^k, ..., a_n^k)$ we have a new one denoted as $(a_1^{k+1}, a_2^{k+1}, ..., a_n^{k+1})$. Thus, for every integer $i = 1, 2, ..., n$, we need to prove

$$\lim_{k \to \infty} a_i^k = a, \quad a = \frac{a_1 + a_2 + ... + a_n}{n}.$$

Let $m_k = \min(a_1^k, a_2^k, ..., a_n^k)$ and $M_k = \max(a_1^k, a_2^k, ..., a_n^k)$.

Clearly, the transformations Δ don't make the value of M_k increase or the value of m_k decrease. Because both m_k and M_k are bound, there exist

$$m = \lim_{k \to \infty} m_k \quad , \quad M = \lim_{k \to \infty} M_k.$$

We must prove that $m = M$. By contradiction, suppose that $M > m$. Denote $d_k = M_k - m_k$. We make a simple observation

Lemma. *Suppose that after carrying out some transformations Δ, the sequence $(a_1^1, a_2^1, ..., a_n^1)$ turns into the new one $(a_1^k, a_2^k, ..., a_n^k)$ satisfying that $m_k = \dfrac{M_1 + m_1}{2}$, then we will have $m_2 = \dfrac{M_1 + m_1}{2}$.*

Indeed, without loss of generality may assume that $M_1 = a_1^1 \geq a_2^1 \geq ... \geq a_n^1 = m_1$.

To be more brief, replace a_i by a_i^1. If $m_k = \dfrac{a_1 + a_n}{2}$ and k is the smallest index satisfying that equality then $a_i^2 \geq m_k \ \forall i \in \{1, 2, ..., n\}$. It follows from $\{m_k\}$ is a non-decreasing sequences. Note that $\dfrac{a_1 + a_n}{2}$ is a term of the sequence $(a_1^2, a_2^2, .., a_n^2)$, so we have done.

By the above property, we have a more important result. Denote

$$S = \{k : \exists l > k | m_k + M_k = 2m_l\} \ \Rightarrow S = \{k | m_k + M_k = 2m_{k+1}\},$$
$$P = \{k : \exists l > k | m_k + M_k = 2M_l\} \ \Rightarrow P = \{k | m_k + M_k = 2M_{k+1}\}.$$

If S or P has an infinite number of elements, assume $|S| = \infty$ then, for each $k \in S$

$$d_{k+1} = M_{k+1} - m_{k+1} = M_{k+1} - \frac{m_k + M_k}{2} \leq \frac{M_k - m_k}{2} = \frac{d_k}{2},$$

because $(d_r)_{r=1}^{+\infty}$ is a decreasing sequence. Thus, if $|S| = \infty$ then $\lim_{r \to \infty} d_r = 0$ and hence $M = m$, the conclusion follows.

Otherwise, we must have $|S|, |P| < +\infty$. Hence, we can suppose that $|S| = |P| = 0$ without affecting the result of the problem. Then, for every $k > 1$ the number $\dfrac{a_1 + a_n}{2}$ can't be the smallest or greatest number in the sequence $(a_1^k, a_2^k, ..., a_n^k)$. So we can consider the confined problem with $n - 1$ numbers when we rejected exactly *one* number $(a_1 + a_n)/2$. By a simple induction method, we have the desired result. ❑

From the above lemma, we have the direct result

Theorem 1 (Stronger mixing variables - S.M.V. theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous, symmetric, under-limitary function satisfying*

$$f(a_1, a_2, ..., a_n) \geq f(b_1, b_2, ..., b_n),$$

in which $(b_1, b_2, ..., b_n)$ is a sequence obtained from the sequence $(a_1, a_2, ..., a_n)$ by the transformation Δ, then we always have

$$f(a_1, a_2, ..., a_n) \geq f(a, a, ..., a)$$

with
$$a = \frac{a_1 + a_2 + ... + a_n}{n}.$$

By this theorem, when using the mixing variables method, we only need to choose the smallest and greatest numbers to perform. By using elementary knowledge, the old mixing variables theorem is proved and improved to have a stronger result. So it can be applied freely.

Moreover, the transformation Δ can be different. For example, we can change it to $\sqrt{ab}$, $\sqrt{\dfrac{a^2 + b^2}{2}}$ or any arbitrary average form. Depending on the supposition of problem, we can choose a suitable way of mixing variables.

2. S.M.V. theorem and some applications

If have never tried proving a difficult inequality which involves more than three variables, it's not easy to understand the importance and significance of S.M.V. theorem. The most useful application of S.M.V. theorem is for four-variable inequalities. Most of the four-variable inequalities are solved more easily by this theorem.

For example, with a familiar problem in IMO shortlist, and the solution is very brief

Problem 1. *Suppose that a, b, c, d are non-negative real numbers whose sum is* 1. *Prove the inequality*

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd.$$

(*Nguyen Minh Duc, IMO Shortlist 1997*)

Solution. Now, we'll consider the most important content of this writing, this is the stronger mixing variables method, or S.M.V. theorem. Without loss of generality, assume that $a \leq b \leq c \leq d$. Denote

$$f(a, b, c, d) = abc + bcd + cda + dab - \frac{176}{27}abcd$$
$$f(a, b, c, d) = ac(b + d) + bd\left(a + c - \frac{176}{27}ac\right).$$

From the supposition, we refer that $a + c \leq \frac{1}{2}(a + b + c + d) = \frac{1}{2}$, hence

$$\frac{1}{a} + \frac{1}{c} \geq \frac{4}{a + c} \geq 8 \geq \frac{176}{27} \Rightarrow f(a, b, c, d) \leq f\left(a, \frac{b + d}{2}, c, \frac{b + d}{2}\right).$$

Considering the transformation Δ for (b, c, d) and as the proved result, we obtain

$$f(a, b, c, d) \leq f(a, t, t, t) \quad , \quad t = \frac{b + c + d}{3}.$$

Now, the problems becomes, if $a + 3t = 1$ then

$$3at^2 + t^3 \leq \frac{1}{27} + \frac{176}{27}at^3.$$

But it's quite simple. Replacing a by $1 - 3t$, we have an obviously true inequality

$$(1 - 3t)(4t - 1)^2(11t + 1) \geq 0.$$

and the conclusion follows immediately. The equality occurs if $a = b = c = d = \frac{1}{4}$ or $a = b = c = \frac{1}{3}, d = 0$ up to permutation. ❏

Return to the introduced inequality *Turkevici*. To my knowledge, all the ways of proving this inequality are complicated or too long. By using S.M.V. theorem in the same manner as in example 3.1.14, it turns out to be very easy.

Problem 2. *Prove the below inequality for all positive real numbers a, b, c, d*

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2.$$

(*Turkevici's Inequality*)

Solution. Assume $a \geq b \geq c \geq d$. Denote

$$\begin{aligned} f(a, b, c, d) &= a^4 + b^4 + c^4 + d^4 + 2abcd - a^2b^2 - b^2c^2 - c^2d^2 - d^2a^2 - a^2c^2 - b^2d^2 \\ &= a^4 + b^4 + c^4 + d^4 + 2abcd - a^2c^2 - b^2d^2 - (a^2 + c^2)(b^2 + d^2). \end{aligned}$$

Hence

$$f(a, b, c, d) - f(\sqrt{ac}, b, \sqrt{ac}, d) = (a^2 - c^2)^2 - (b^2 + d^2)(a - c)^2 \geq 0.$$

By S.M.V. theorem with the transformation Δ of (a, b, c), we only need to prove the inequality when $a = b = c = t \geq d$. In this case, the problem becomes

$$3t^4 + d^4 + 2t^3d \geq 3t^4 + 3t^2d^2 \Leftrightarrow d^4 + t^3d + t^3d \geq 3t^2d^2.$$

By $AM - GM$ Inequality, the problem is obviously true. The equality is taken if and only if $a = b = c = d$ or $a = b = c, d = 0$ or permutations. ❑

Problem 3. *Let x, y, z, t be positive numbers satisfying the condition $x+y+z+t = 4$. Prove that*

$$(1 + 3x)(1 + 3y)(1 + 3z)(1 + 3t) \leq 125 + 131xyzt.$$

(*Pham Kim Hung*)

Solution. It's easy to check that the equality occurs if $x = y = z = t = 1$ or $x = y = z = 4/3, t = 0$. So 131 is the greatest value of k for the following inequality

$$(1 + 3x)(1 + 3y)(1 + 3z)(1 + 3t) \leq 256 + k(xyzt - 1).$$

Consider the expression

$$f(x, y, z, t) = (1 + 3x)(1 + 3y)(1 + 3z)(1 + 3t) - 131xyzt.$$

Without loss of generality, we may assume $x \geq y \geq z \geq t$. Hence

$$\begin{aligned} &f(x, y, z, t) - f\left(\frac{x+z}{2}, y, \frac{x+z}{2}, t\right) \\ &= 9(1 + 3y)(1 + 3t)\left(xz - \frac{(x+z)^2}{4}\right) - 131yt\left(xz - \frac{(x+z)^2}{4}\right). \end{aligned}$$

Note that if $y + t \leq 2$ then

$$9(1 + 3y)(1 + 3t) \geq 131yt \Leftrightarrow 9 + 27(y + t) \geq 50yt.$$

Because $y+t \le 2$, thus $yt \le 1$, Hence

$$9+27(y+t) \ge 54\sqrt{yt} \ge 54yt \ge 50yt$$

which yields that $f(x,y,z,t) \le f\left(\frac{x+z}{2}, y, \frac{x+z}{2}, t\right)$. By $S.M.V$ theorem, it's enough to prove the inequality in case $x=y=z=a \ge 1 \ge t = 4-3z$ and in that case, we obtain

$$(1+3a)^3(1+3(4-3a)) \le 125 + 131a^3(4-3a).$$

After expending and collecting terms, the above inequality becomes

$$150a^4 - 416a^3 + 270a^2 + 108a - 112 \le 0$$

$$\Leftrightarrow (a-1)^2(3a-4)(50a+28) \le 0,$$

which is clearly true. We have equality if $a=1$ or $a=4/3$, which is equivalent to the two cases of equality showed at the beginning of solution. ❑

The below problem is full of the color and character of this method.

Problem 4. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers satisfying that $a_1a_2...a_n = 1$. Prove that*

$$\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} + \frac{3n}{a_1+a_2+...+a_n} \ge n+3,$$

for all positive integer $n \ge 4$.

(*Pham Kim Hung*)

Solution. Without loss of generality, we may assume that $a_1 \ge a_2 \ge ... \ge a_n$. Denote

$$f(a_1,a_2,...,a_n) = \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} + \frac{3n}{a_1+a_2+...+a_n},$$

We will prove that

$$f(a_1,a_2,...,a_n) \ge f(a_1, \sqrt{a_2a_n}, \sqrt{a_2a_n}, a_3, a_4, ..., a_{n-1}) \quad (*)$$

Indeed, this one is equivalent to

$$\begin{aligned} &f(a_1,a_2,...,a_n) - f(a_1,\sqrt{a_2a_n},\sqrt{a_2a_n},a_3,a_4,...,a_{n-1}) \\ &= \left(\frac{1}{\sqrt{a_2}} - \frac{1}{\sqrt{a_n}}\right)^2 - \frac{3n(\sqrt{a_2}-\sqrt{a_n})^2}{(a_1+a_2+...+a_n)(a_1+2\sqrt{a_2a_n}+a_3+...+a_{n-1})}, \end{aligned}$$

hence, it suffices to prove that

$$(a_1+a_2+...+a_n)(a_1+2\sqrt{a_2a_n}+a_3+...+a_{n-1}) \ge 3na_2a_n.$$

Because $a_1 \geq a_2 \geq ... \geq a_n$, we deduce that

$$\begin{aligned}
&(a_1 + a_2 + ... + a_n)(a_1 + 2\sqrt{a_2 a_n} + a_3 + ... + a_{n-1}) \\
&\geq (2a_2 + (n-2)a_n)(a_2 + 2\sqrt{a_2 a_n} + (n-3)a_n) \\
&\geq (2 + 2\sqrt{n-3})2\sqrt{2(n-2)}a_2 a_n \geq 3n a_2 a_n,
\end{aligned}$$

for all non-negative integer $n \geq 4$. Otherwise, for $n = 3$, this one is still true (with $a_2 \geq a_3$) because

$$(a_2 + 2\sqrt{a_2 a_3} + a_3)(2a_2 + a_3) \geq 9a_2 a_3.$$

Thus $(*)$ is proved.
Furthermore, $(*)$ brings us a more important result. By S.M.V. theorem, we have

$$f(a_1, a_2, ..., a_n) \geq f(a_1, b, b, ..., b), \quad b = \sqrt[n-1]{a_2 a_3 ... a_n},$$

and the rest is (before replacing n by $n+1$ for aesthetics) proving that $g(b) \geq n+4$.

$$\begin{aligned}
g(b) &= b^n + \frac{n}{b} + \frac{3(n+1)}{nb + 1/b^n} = b^n + \frac{n}{b} + \frac{3(n+1)b^n}{nb^{n+1} + 1} \\
g'(b) &= nb^{n-1} - \frac{n}{b^2} + \frac{3(n+1)(nb^{n-1}(nb^{n+1} + 1) - (n+1)nb^{2n})}{(nb^{n+1} + 1)^2} \\
g'(b) &= nb^{n-1} - \frac{n}{b^2} + \frac{3n(n+1)}{(nb^{n+1} + 1)^2}(b^{n-1} - b^{2n}).
\end{aligned}$$

Hence

$$g'(b) = 0 \Leftrightarrow (b^{n+1} - 1)((nb^{n+1} + 1)^2 - 3(n+1)b^{n+1}) = 0.$$

By $AM - GM$ Inequality, we get $(nb^{n+1} + 1)^2 \geq 4nb^{n+1} \geq 3(n+1)b^{n+1}$.
Thus $g'(b) \leq 0 \ \forall b \leq 1$ and $g'(1) = 0$, imply $g(b) \geq g(1) = n + 4$, which is exactly the desired result. The equality holds for $a_1 = a_2 = ... = a_n = 1$. ❑

Notice that in the above problem, the best constant to replace $3n$ is $4(n-1)$, hence we need to add the condition $n \geq 4$. But the solution for each one is similar.

Problem 5. *Let a, b, c, d be positive real numbers adding up to 4 and k is a given positive real number. Find the maximum value of the expression*

$$(abc)^k + (bcd)^k + (cda)^k + (dab)^k.$$

(*Pham Kim Hung*)

SOLUTION. Without loss of generality, we may assume $a \geq b \geq c \geq d$. Firstly, suppose that $1 \leq k \leq 3$. Let $t = \frac{a+c}{2}$ and $u = \frac{a-c}{2}$, we get $a = t + u, c = t - u$.

$$(abc)^k + (bcd)^k + (cda)^k + (dab)^k = (b^k + d^k)(ac)^k + b^k d^k (a^k + c^k).$$

Let $s = b^{-k} + d^{-k}$ and consider the function

$$f(u) = s(ac)^k + a^k + c^k = s(t^2 - u^2)^k + (t+u)^k + (t-u)^k.$$

We will prove that $f(u) \le f(0)$. Indeed

$$\begin{aligned} f'(u) &= -2kus(t^2-u^2)^{k-1} + k(t+u)^{k-1} - k(t-u)^{k-1} \\ &= ku(t^2-u^2)^{k-1}\left(-2s + \frac{(t-u)^{-k+1} - (t+u)^{-k+1}}{2u}\right). \end{aligned}$$

Because $a \ge b \ge c \ge d$, so $d \le t - u$. On the other hand, because $k \le 3$ and $\delta(x) = x^{-k+1}$ is a decreasing function, so *Lagrange* Theorem implies that

$$\begin{aligned} \frac{(t+u)^{-k+1} - (t-u)^{-k+1}}{2u} &= \delta'(\beta) \ge (-k+1)(t-u)^{-k} \\ \Rightarrow \frac{(t-u)^{-k+1} - (t+u)^{-k+1}}{2u} &\le (k-1)(t-u)^{-k} \\ \Rightarrow -2s + \frac{(t-u)^{-k+1} - (t+u)^{-k+1}}{2u} &\le \frac{-2}{d^k} + \frac{k-1}{(t-u)^k} \le 0. \end{aligned}$$

Thus $f(u) \le f(0)$. By S.M.V. Theorem, we only need to prove the problem in case $a = b = c = t \ge d$. Consider the function

$$g(t) = t^{3k} + 3t^{2k}(4-3t)^k,$$

We will prove $g(t) \le \max\left(g(1), g(\frac{4}{3})\right)$. Indeed,

$$g'(t) = 3kt^{3k-1} + 6kt^{2k-1}(4-3t)^k - 9kt^{2k}(4-3t)^{k-1}$$

$$\begin{aligned} g'(t) = 0 &\Leftrightarrow t^k + 2(4-3t)^k = 3t(4-3t)^k \\ &\Leftrightarrow \left(\frac{t}{4-3t}\right)^k + 2 = \frac{3t}{4-3t}. \end{aligned}$$

Let $r = r(t) = \dfrac{t}{4-3t} \Rightarrow r(t)$ is a monotonically increasing function, that yields

$$g'(t) = 0 \Leftrightarrow r^k + 2 = 3r.$$

Clearly, the above equation has no more than two positive real roots. Since $g'(1) = 0$, we deduce that

$$g(t) \le \max\left(g(1), g\left(\frac{4}{3}\right)\right) = \max\left(4, \left(\frac{4}{3}\right)^{3k}\right).$$

From the above result, we find out (by taking $k = 1$)

$$abc + bcd + cda + dab \le 4,$$

Hence for all $k \leq 1$ then

$$(abc)^k + (bcd)^k + (cda)^k + (dab)^k \leq 4.$$

Consider the case $k \geq 3$, we have

$$(abc)^k + (bcd)^k + (cda)^k + (dab)^k \leq (ab)^k(c+d)^k$$
$$\Leftrightarrow (ab)^k \left((c+d)^k - c^k - d^k\right) \geq (a^k + b^k)c^k d^k.$$

This last inequality is obviously true because $(c+d)^k - c^k - d^k \geq kc^{k-1} \geq 2c^k$. Moreover, applying the $AM - GM$ Inequality

$$(ab)^k(c+d)^k \leq \left(\frac{a+b+c+d}{3}\right)^{3k} = \left(\frac{4}{3}\right)^{3k}.$$

Hence the inequality has been proved completely

$$(abc)^k + (bcd)^k + (cda)^k + (dab)^k \leq \max\left(4, \left(\tfrac{4}{3}\right)^{3k}\right). \quad ❏$$

To conclude try applying the method to the following examples to improve your skill.

Problem 6. *Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 4$. Prove the following inequality*

$$16 + 2abcd \geq 3(ab + ac + ad + bc + bd + cd).$$

Problem 7. *Let $a, b, c, d, e \geq 0$ satisfy that $a + b + c + d + e = 5$. Prove that*

$$4(a^2 + b^2 + c^2 + d^2 + e^2) + 5abcde \geq 25.$$

Problem 8. *Let a, b, c, d be non-negative real numbers and $a + b + c + d = 4$. Prove that*

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (1 + a)(1 + b)(1 + c)(1 + d).$$

(*Pham Kim Hung*)